11 Functions

71. **(2nd exam, January 2022)** The function $f : \mathbb{Z} \to \mathbb{Z}$ is defined by

$$f(x) = x^2.$$

Examine the injectivity, surjectivity, and bijectivity of the function f.

72. Let $\mathbb{R}$ be the set of real numbers, and let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by

$$f(x) = 5x + 12.$$

Examine the injectivity, surjectivity, and find $f^{-1}(x)$.

73. Consider the following relation:

$$\{(x, y) \in \mathbb{R} \times \mathbb{R} : y^2 = \sqrt{x}\}.$$

Determine if the given relation is a function on $\mathbb{R}$. (Provide a detailed explanation of your answer.)

74. Let $\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\}$. Consider the following relation:

$$\{(x, y) \in \mathbb{R}_0^+ \times \mathbb{R} : y^2 = \sqrt{x}\}.$$

Determine if the given relation is a function from $\mathbb{R}_0^+$ to $\mathbb{R}$. (Provide a detailed explanation of your answer.)

75. Let $\mathbb{R}_0^+ = \{x \in \mathbb{R} \mid x \geq 0\}$. Consider the following relation:

$$R = \{(x, y) \in \mathbb{R}_0^+ \times \mathbb{R}_0^+ : y^2 = \sqrt{x}\}.$$

Determine if the given relation is a function on $\mathbb{R}_0^+$. (Provide a detailed explanation of your answer.)

76. Let $a, b \in \mathbb{R}$ and $b \neq 0$. The function $f : \mathbb{R} \setminus \{0\} \to \mathbb{R} \setminus \{a\}$ is defined by

$$f(x) = a + \frac{b}{x}.$$

Examine the injectivity, surjectivity, and bijectivity of the function f.

77. Find an example of:

(a) An injective function $f : \mathbb{N} \to \mathbb{N}$ that is not surjective.

(b) A function $f : \mathbb{N} \to \mathbb{N}$ that is surjective but not injective.

78. Let $\mathbb{N}$ be the set of natural numbers, and let $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function defined by

$$g(m, n) = 2^m \cdot 3^n.$$

Examine the injectivity, surjectivity, and determine g^{-1} (if it exists).

79. Let $\mathbb{Z}$ be the set of integers, and let $f : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ be a function defined by

$$f(m, n) = m + n.$$

Examine the injectivity, surjectivity, and determine f^{-1} (if it exists).

80. Let $f : A \to B$ be a function, and let $\text{im}(f) := \{y \in B \mid (\exists x)((x, y) \in f)\}$. Prove the following implication:

- If f is an injective function, then f^{-1} is a function from $\text{im}(f)$ to A.

(In the above implication, f^{-1} is the inverse relation of the relation f, i.e., $f^{-1} = \{(y, x) \mid (x, y) \in f\}$.)

81. **(Theoretical Exercise)** Let $f : A \to B$ be a function. If there exists a function $g : B \to A$ such that

$$f \circ g = \text{id}_B \qquad \text{and} \qquad g \circ f = \text{id}_A,$$

show that f is a bijective function and $g = f^{-1}$.

82. **(Theoretical Exercise)** Let $f : A \to B$ and $g : B \to A$ be given functions. Prove the following implications:

- If $f \circ g = \text{id}_B$, then $f \circ g$ is a surjective function.
- If $f \circ g$ is a surjective function, then the function f is also surjective.

(In the first implication above, id_B is the identity function on the set B, i.e., $\text{id}_B(b) = b$ for all $b \in B$.)

All above math problems are taken from the following website:
`https://osebje.famnit.upr.si/~penjic/teaching.html`.
The reader can find all solutions to the given problems on the same page.